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Quantum localization and cantori in the stadium billiard

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We discuss the quantum dynamics of the stadium billiard whose classical motion is known to be completely chaotic. In particular, we show how the simultaneous presence of cantori in the classical phase space and of the phenomenon of quantum dynamical localization affects the structure of the eigenfunctions and the statistical properties of the eigenvalues. [S1063-651X(99)50203-1]

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The study of quantum mechanics of complex systems in light of the chaotic behavior of the corresponding classical systems has greatly improved our understanding of quantum motion [1]. For example, the possibility put forward long ago [2] that random matrix theory (RMT) may be a convenient tool to describe spectral properties of classically chaotic systems rests now on more solid ground [3,4]. However, in spite of the progress in recent years and the growing interest in the so-called ''quantum chaos'' we are still far from a satisfactory understanding, as a great variety and, to some extent, unexpected rich behavior of quantum motion continues to emerge [5] for which a satisfactory explanation is required. For example, the phenomenon of quantum dynamical localization discovered 20 years ago in systems under external perturbations [6] and now experimentally confirmed [7], mainly rests on numerical computations and on qualitative considerations. Only a few mathematical results exist and it is not clear whether existing semiclassical theories can account for this important feature. Even less understood is the mechanism of dynamical localization in conservative systems [8]. Billiards are very convenient models to study since they can display a rich variety of dynamical behavior from completely integrable (e.g., the circle) to weakly mixing (e.g., the right triangle with irrational angle) [9], to completely chaotic with power law decay of correlations (the stadium), up to exponential decay of correlations (dispersive billiards). In addition, they can be studied numerically with great efficiency and high accuracy (here we are able to compute accurate eigenvalues and eigenfunctions of the *desymmetrized* billiard with sequential number up to 10^7). They can be, to some extent, studied analytically and in laboratory experiments [10]. Also, they may be relevant for technological applications such as the design of novel microlasers or other optical devices [11].

Recently, localization has been shown to take place in the stadium billiard [12] and in other similar models [13–16]. It is associated with the fact that, for small perturbations of the circle, the angular momentum undergoes a classical diffusive process and quantum effects may lead to suppression of this diffusive excitation.

The rich variety of classical phase space determines a quite complicated quantum structure. Indeed, the classical motion in the stadium billiard can be described by a discontinuous map of the saw-tooth type. This map is known to have cantori [17], which may act as barriers to quantum motion [18]. This effect has been discussed in [19] and recently confirmed in numerical computations on the saw-tooth map on the cylinder [15].

In the following we discuss how the combined presence of the cantori structure and of quantum dynamical localization acts on eigenfunctions (EF) until the regime of quantum ergodic behavior is reached. We consider the motion of a free point particle of unit mass and velocity $\vec{v}$ (energy $E=v^2/2$) bouncing elastically inside a stadium-shaped well: two semicircles of radius 1 connected by two straight line segments of length 2ϵ. The classical motion, for arbitrary small ϵ, is ergodic, mixing, and exponentially unstable with Lyapunov exponent $\Lambda\sim\epsilon^{1/2}$. It can be approximated [up to $\mathcal{O}(\epsilon)$] by the discontinuous stadium-map [12] $L_{n+1}=L_n-2\epsilon\sin\theta_n\,\mathrm{sgn}(\cos\theta_n)\sqrt{1-L_n^2}$; $\theta_{n+1}=\theta_n+\pi-2\arcsin L_{n+1}$ for the rescaled angular momentum $L=l/\sqrt{2E}$, where $l=\vec{r}\wedge\vec{v}$, and for the polar angle θ (identical to arc length for small ϵ). From rigorous results on the saw-tooth map [17] and from the stadium dynamics it can be shown that the angular momentum (for small ϵ) undergoes a normal diffusive process with diffusion rate

$$D=\langle(l_n-l_0)^2\rangle/n|_{n\gg1}\approx2\epsilon^{5/2}(2E-l^2). \qquad (1)$$

(Notice that the diffusion rate D depends on the local value of angular momentum.) The power 5/2 in Eq. (1) is due to the existence of *cantori*, which form strong obstacles to phase space transport. In Ref. [12] the phenomenon of quantum localization has been shown to take place in the stadium billiard leading to strong deviations from RMT predictions. However, the dependence of the localization length on system parameters is not known and in particular we do not know if, and to what extent, the presence of cantori will influence the quantum dynamics. Indeed it has been conjectured [19] that cantori act as perfect barriers for quantum motion provided the flux through cantori is smaller than a Planck's cell, $\mathcal{F}<2\pi\hbar$. On the basis of results on the saw-tooth map [17] we can estimate the *flux* $\mathcal{F}$ [the phase space area transported through cantori per iteration (bounce with

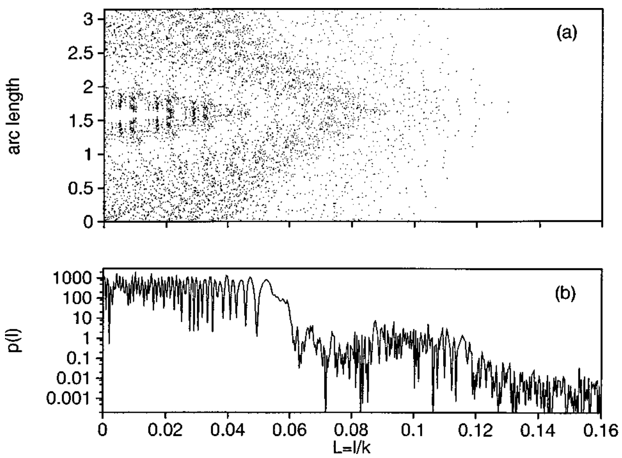

FIG. 1. (a) A single classical orbit, followed up to 20 000 bounces, for the billiard with $\epsilon=0.003$. The orbit is started in the middle of the largest ''island'' ($L=0$, $\theta=\pi/4-0.0016$). (b) Angular momentum probability distribution $p_k(l)$ of the corresponding eigenstate Ψ_k with eigenvalue $k=5999.8166$. As it is seen, the state is uniformly distributed over the cantorus in the main island.

the boundary)] which is here independent of the winding number of the resonance and, for small ϵ, it is given by $\mathcal{F}\approx(2E)^{1/2}\epsilon^{3/2}$, which leads to the *cantori border*

$$\epsilon_c=k^{-2/3}, \tag{2}$$

where $k=\sqrt{2E}$ is the wave number. For $\epsilon<\epsilon_c$ ($x:=k\epsilon^{3/2}<1$), cantori act as perfect barriers, and the quantum system looks as if it is classically integrable. It is therefore expected that the localization length ℓ of eigenstates in angular momentum variable l must be of the order of the size of cantori. This size, in rescaled angular momentum variables, averaged over all the resonances, can be estimated from the exact results on saw-tooth map [20], namely $\overline{p}=c\epsilon$, where $c\approx12$ is a numerical constant ($c=10$ for $\epsilon=0.05$, and $c=15$ for $\epsilon=0.005$). The fact that c slowly increases with decreasing ϵ is due to the presence of the cantorus along the separatrix of 2:1 resonance (around $L=0$) which has a larger size, $p(2,1)\approx\sqrt{\epsilon}$. In Fig. 1 we show the classical structure of cantori ($\epsilon=0.003$) in phase space around the largest 2:1 resonance and the associated quantum eigenstate. In this regime the (average) rescaled localization length of eigenstates $\sigma=\ell/\ell_{\max}=\ell/k$ is indeed found to be equal to the (average) size of cantori (see Fig. 2), $\sigma=\overline{p}$.

For $\epsilon>\epsilon_c$ ($x=k\epsilon^{3/2}>1$), when the flux trough turnstiles becomes larger than one quantum $2\pi\hbar$, the cantori do not act any more as barriers for quantum dynamics and the quantum motion follows the classical diffusive behavior up to the quantum relaxation time t_R (break time), which is proportional to the density of *operative eigenstates*. For $t>t_R$ instead, the quantum dynamics enters an oscillatory regime around the stationary localized state with a localization length ℓ. Therefore, $t_R=\sigma2\pi dN/dE$, where $\sigma=\ell/\ell_{\max}=\ell/k$, and $N=E/8$ (from the Weyl formula). From the diffusion law (1) we have $\ell^2\approx Dt_R/T\approx\sigma D/T$, where $T\approx E^{-1/2}$ is the average time between bounces. Thus we obtain a simple expression for the rescaled, averaged localization length σ,

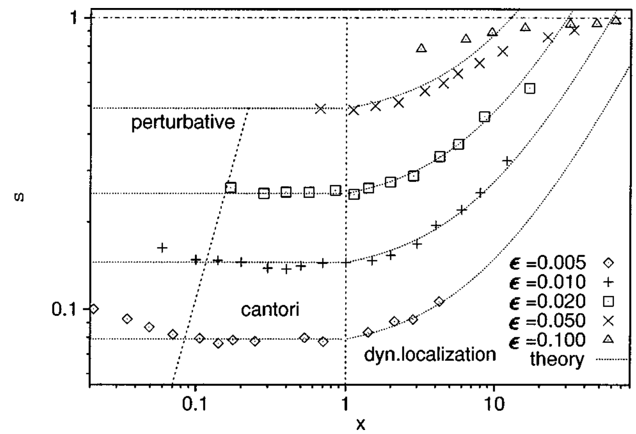

FIG. 2. Rescaled localization length σ vs the scaling variable $x=\epsilon^{3/2}k$ for five values of ϵ ($60<k<12{,}000$). Each point is obtained by averaging over a large number ν of consecutive eigenstates ($\nu=100$ for small k and $\nu=1{,}000$ for large k). The numerical data clearly show the cantori border $x=1$. In the cantori region σ is constant as expected, while for $x>1$ the numerical data agree with the theoretical prediction (4) (dotted curves). For large x, the value of σ approaches the maximal ergodic value $\sigma=1$.

$$\sigma\approx D/k\approx\alpha\epsilon^{5/2}k=\alpha\epsilon x, \tag{3}$$

where $\alpha\approx1.7$ is a numerical constant. However, we need to add the average size $\overline{p}$ of cantori to the above expression. Therefore, for $x>1$, the actual expression for the localization length will be given by

$$\sigma=\overline{p}+(1-\overline{p})\alpha\epsilon(x-1), \tag{4}$$

which takes into account also the fact that we need to rescale the total size of angular momentum space, and that for $x=1$, $\sigma=\overline{p}$. Eigenstates become delocalized (ergodic) when $\sigma=1$; this defines the ergodicity border,

$$\epsilon_e\approx(\alpha k)^{-2/5}, \tag{5}$$

in agreement with the results of [12]. The *cantori border* can actually be observed if it is below the ergodic border and above the perturbative border ϵ_p. The *perturbative border* ϵ_p is given by the condition that ϵ should be large enough, $\epsilon>\epsilon_p\approx k^{-1}$, to couple two neighboring eigenvalues of angular momentum, which is equivalent to the intuitive condition of comparing the deformation ϵ with the de Broglie wavelength. Therefore, for sufficiently large k, we have $\epsilon_p<\epsilon_c<\epsilon_e$. In this situation it is natural to expect that cantori will influence the localization process and we may have here a nice possibility to study the effect of cantori in quantum mechanics.

In order to check the above predictions we numerically computed quantum eigenfunctions $\Psi_k(\vec{r})$ of the stadium billiard [solutions of the Schrödinger equation $(\nabla^2+k^2)\Psi_k=0$, where $\hbar=1$] by expanding them in terms of circular waves (here we consider only odd-odd states) $\Psi_k(\vec{r})=\sum_{s=1}^{M}a_sJ_{2s}(kr)\sin(2s\theta)$. The eigenvalues $k=k_n$ and the associated coefficients a_s have been computed very efficiently [16] by minimizing a special quadratic form defined along the boundary of the billiard [21]. The coefficient a_s is pro-

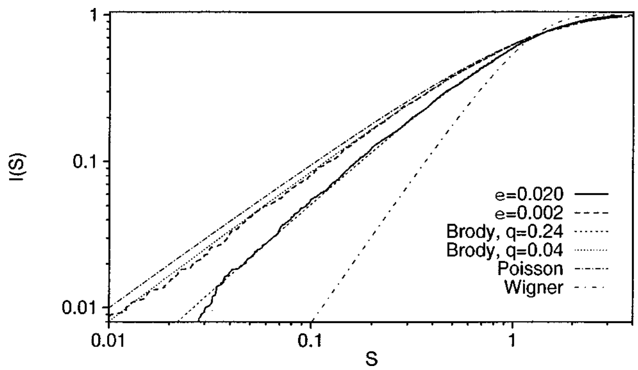

FIG. 3. Integrated level spacing distribution $I(S)=\int_0^S ds P(s)$ for two spectral samples with $k\sim 2000$ and $\epsilon=0.002$ (cantori regime), $\epsilon=0.02$ (dynamical localization regime). The samples contain about 3500 consecutive energy levels each. We also plot the best fitting Brody distributions $I_B(S)=1-\exp(-aS^{q+1})$ with $q=0.24$, and $q=0.04$ (nearly Poisson).

portional to the probability amplitude of finding angular momentum equal to $l=2s$, $p_k(l)=|\langle l=2s|\Psi_k\rangle|^2 \propto |a_s|^2\sqrt{1-l^2/k^2}$.

Quantum localization in the stadium is not exponential as for the kicked rotor or smooth diffusive billiards [13]. Indeed the tails of eigenfunctions have been found [16] to decay, on average, as $p(l)\sim|l-\langle|l|\rangle|^{-4}$. Therefore, in order to characterize the localization length of quantum eigenfunctions Ψ_k, we choose the 99% probability localization length σ rather than the more common inverse participation ratio or information entropy, etc. Indeed, the former is quite independent of the nature of tails of the distribution $p(l)$. More precisely, we define σ_k as the minimal number of angular momentum states that are needed to support the 99% probability of an eigenstate Ψ_k, $\sigma_k=\min\{\#\mathcal{A},\Sigma_{l\in\mathcal{A}}p_k(l)\geqslant 0.99\}/(0.725k)$. The normalization factor has been chosen in such a way that $\sigma_k=1$ for completely delocalized (GOE) states. In Fig. 2 we show the dependence of the averaged, rescaled localization length $\sigma=\langle\sigma_k\rangle$ (averaged over a set of consecutive eigenstates Ψ_k with k in a narrow interval) on the parameter ϵ and on the wave number k (up to $k=12\,000$, $N\approx 10^7$). Numerical data agree with theoretical predictions (4) with $\alpha=1.7$.

We have also analyzed the deviation of levels statistics from RMT. Below the cantori border, $x<1$, the system acts as if classically integrable and the nearest neighbor levels spacing distribution $P(S)$ is nearly Poissonian, $P(S)\approx\exp(-S)$, while in the regime of true dynamical localization $P(S)$ is intermediate between Poissonian and Wigner-Dyson. Furthermore we have significant numerical evidence for the fractional power-law level repulsion $P(S\to 0)\propto S^q$, which is well approximated by the phenomenological Brody distribution $P(S)=bS^q\exp(-aS^{q+1})$, $b=\Gamma[1+(1+q)^{-1}]^{q+1}$, $a=(q+1)b$ with exponent $0<q<1$; see Fig. 3. However, the aim here is to study localization of eigenstates rather than to find the exact form of $P(S)$, so we use level spacing distribution merely to show deviation from Wiger-Dyson.

The exact stadium eigenstates Ψ_{k_n} may be expanded in terms of unperturbed quarter-circle eigenstates $\Phi_{sm}(\vec{r})=J_{2s}(k^0_{sm}r)\sin(2s\theta)$, where k^0_{sm} are the eigenvalues of the

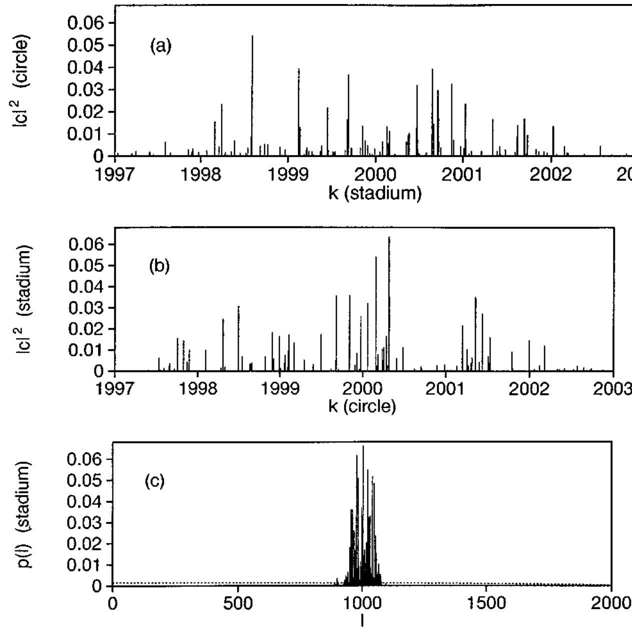

FIG. 4. Structure of a typical eigenstate for $\epsilon=0.01$. (a) Probability distribution of a circle state ($l=848$, $k^0_{sm}=1999.993\,49$) vs the wave number of the stadium eigenstates (a column of the matrix $|c^n_{sm}|^2$ with fixed s,m); (b) probability distribution of a stadium eigenstate ($k_n=1999.913\,97$) vs the wave number of circle states (a row of the matrix $|c^n_{sm}|^2$ with fixed n); (c) the same state as in (b) in angular momentum quantum number l. Here the rescaled localization length is $\sigma_{k_n}=0.10$. The dotted curve in (c) gives the ergodic distribution $p_e(l)=(8\pi/k^2)\sqrt{k^2-l^2}$.

integrable quarter circle (the zeros of the even-order Bessel functions). The matrix $c^n_{sm}=\langle\Phi_{sm}|\Psi_{k_n}\rangle$ can be easily computed from the coefficients a_l [13].

It is important to note that, unlike for Wigner band random matrices [8], the matrix c^n_{sm} (ordered, as usual, with increasing quantum wavenumbers k_n, k^0_{sm}) has a *symmetric* appearance [16]. It has a band, *sparse* structure with bandwidth $b\sim k$, independent of ϵ (for $\epsilon>\epsilon_p$). The effective number of nonzero elements in each row (or column) is (on average) equal to the localization length $\ell=\sigma k$. The (statistical) structure of the rows (stadium eigenstates, fixed n) is typically found to be very similar to the structure of the columns (circle states, fixed s,m). In order to illustrate the general structure we show in Figs. 4(a) and 4(b) the probability distributions of a typical circle state in terms of eigenstates of the stadium and vice versa. These distributions are strongly sparse inside the band $b\sim k$. In Fig. 4(c) we show, for the same stadium eigenstate, the probability distribution $p(l)$ in angular momentum variable l: as expected it is strongly localized and nonsparse. The above structure is found in the regime of dynamical localization and in the regime of cantori localization where the quantum system behaves as if classically integrable [22].

As the parameter ϵ is increased up to the ergodicity border ϵ_e, sparsity decreases and the quantum angular momentum distribution $p(l)$ approaches (apart from fluctuations) the classical steady-state microcanonical distribution, $p_e(l)\propto\sqrt{2E-l^2}$. Notice that the scaling parameter σ also controls the deviations from RMT predictions.

In this paper we have shown that the quantum dynamics of the classically chaotic stadium billiard exhibits a rich structure and different regimes of motion as a function of ϵ and energy E. It has been shown that the presence of *cantori* in classical phase space may have strong effects on the quantum dynamics and leads to a border that is different from the perturbative and ergodic border. In the regime of quantum cantori (where the phase space flux through cantori is less than one quantum) the rescaled localization length $\sigma = \ell / k$ does not depend on energy or wave number $k = \sqrt{2E}$. However, above the cantori border, quantum dynamical localization takes place and the localization length ℓ is found to be proportional to the rate D of classical diffusion in angular momentum. The mechanism of localization is strongly connected to the sparsity of EFs when expanded on the basis of (unperturbed) circle states (and vice versa). We would like to mention that we have also analyzed another conservative model exhibiting classical diffusive behavior, namely, the rough billiards [13] (having analytic, nearly circular but randomly wiggled boundary), where we have found very similar results for the structure of the matrix c^n_{sm}: symmetric structure of rows and columns, and strong sparsity for $\epsilon < \epsilon_e$. We suggest that the above features are typical of the quantum dynamics of conservative systems for which the Hamiltonian can be written as a sum of an integrable part plus a small perturbation, which renders the total Hamiltonian completely chaotic. In this case one expects a behavior of the type described here, including classical diffusion in unperturbed action variables, quantum localization, etc.

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